10.) $X^3+125=0$

possible # real roots = 3 or 1

Possible rational roots: $\pm 1, \pm 5, \pm 25, \pm 125$

Sum of cubes $(a+b)^3 = (a+b)(a^2-ab+b^2)$

$X^3+125=0$

$(x+5)(x^2-5x+25)=0$

↑ Need to use quadratic equation

$X+5=0$ or $X = \dfrac{5 \pm \sqrt{(-5)^2-4(1)(25)}}{2(1)}$

$X=-5$

$X = \dfrac{5 \pm \sqrt{25-100}}{2}$

$X = \dfrac{5 \pm \sqrt{-75}}{2}$

$X = \dfrac{5 \pm 5i\sqrt{3}}{2}$ ← Provides 2 imaginary roots

So roots are $\left\{-5, \dfrac{5+5i\sqrt{3}}{2}, \dfrac{5-5i\sqrt{3}}{2}\right\}$

11. $X^3 - 8 = 0$

\# of real roots = 3 or 1

Possible rational roots: $\pm 1, \pm 2, \pm 4, \pm 8$

* Difference of cubes $(a-b)^3 = (a-b)(a^2+ab+b^2)$

$$X^3 - 8 = 0$$

$$(X-2)(X^2+2X+4) = 0$$

$$X = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Need to use quadratic formula. Why?

$(X-2)=0$ OR

$X = 2$

$$X = \frac{-2 \pm \sqrt{2^2-4(1)(4)}}{2(1)}$$

$$X = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$X = \frac{-2 \pm \sqrt{-12}}{2}$$

$$X = \frac{-2 \pm 2i\sqrt{3}}{2}$$

$X = -1 \pm i\sqrt{3}$ ← provides 2 imaginary roots

So roots are $\{2, -1+i\sqrt{3}, -1-i\sqrt{3}\}$

(12.) $X^4 + X^2 - 12 = 0$

possible # Real roots : 4, 2, or 0

Possible rational roots: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Test
$$\begin{array}{r|rrrrr} \cancel{3} & 1 & 0 & 1 & 0 & -12 \\ & \downarrow & 3 & 9 & 30 & 90 \\ \hline & 1 & 3 & 10 & 30 & 78 \end{array}$$

$$\begin{array}{r|rrrrr} \cancel{-1} & 1 & 0 & 1 & 0 & -12 \\ & \downarrow & -1 & 1 & -2 & 2 \\ \hline & 1 & -1 & 2 & -2 & \boxed{-10} \end{array}$$

$$\begin{array}{r|rrrrr} \cancel{2} & 1 & 0 & 1 & 0 & -12 \\ & \downarrow & 2 & 4 & 10 & 20 \\ \hline & 1 & 2 & 5 & 10 & \boxed{8} \end{array}$$

$$\begin{array}{r|rrrrr} \cancel{1} & 1 & 0 & 1 & 0 & -12 \\ & \downarrow & 1 & 1 & 2 & 2 \\ \hline & 1 & 1 & 2 & 2 & -10 \end{array}$$

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & 1 & 0 & -12 \\ & \downarrow & -2 & 4 & -10 & 20 \\ \hline & 1 & -2 & 5 & -10 & \boxed{8} \end{array}$$

$$\begin{array}{r|rrrrr} -3 & 1 & 0 & 1 & 0 & -12 \\ & \downarrow & -3 & 9 & & -30 \\ \hline & 1 & -3 & 10 & & \boxed{-42} \end{array}$$

Check something else

$(x^2 + 4)(x^2 - 3) = 0$

$X^2 + 4 = 0$ or $X^2 - 3 = 0$

$X^2 = -4$ $\qquad X^2 = 3$

$X = \pm\sqrt{-4}$ $\qquad X = \pm\sqrt{3}$

$X = \pm 2i$

So in this case, two roots are irrational and two are imaginary.

$X = 2i, -2i, \sqrt{3}, -\sqrt{3}$

(13.) $x^3 + 2x^2 + 3x + 6 = 0$

possible # real roots: 3 or 1

possible rational roots: $\pm 1, \pm 2, \pm 3, \pm 6$

$$\begin{array}{r|rrrr} \cancel{-1} & 1 & 2 & 3 & 6 \\ & \downarrow & -1 & -1 & -2 \\ \hline & 1 & 1 & 2 & 4 \end{array} \qquad \begin{array}{r|rrrr} -2 & 1 & 2 & 3 & 6 \\ & \downarrow & -2 & 0 & -6 \\ \hline & 1 & 0 & 3 & \boxed{0} \end{array}$$

$x^2 + 0x + 3$ so $(x+2)$ is a factor

$x^3 + 2x^2 + 3x + 6$

$0 = (x+2)(x^2+3)$ ← Cannot factor any more.

$x + 2 = 0$ or $x^2 + 3 = 0$

$x = -2$ $\qquad$ $x^2 = -3$

$x = \pm\sqrt{-3}$

$x = \pm i\sqrt{3}$

So roots are $\{-2, i\sqrt{3}, -i\sqrt{3}\}$ 1 real + 2 imaginary

14.) $x^3+1=0$

possible # real roots: 3 or 1

possible rational factors: ± 1

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & \downarrow & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

(x^2-x+1)

$(x+1)$ is a factor

Sum of cubes

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$x^3+1=0$

$$0=(x+1)(x^2-x+1)$$ ← Must use quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{1\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$$

$x+1=0$ OR

$x=-1$

$$x=\frac{1\pm\sqrt{1-4}}{2}$$

$$x=\frac{1\pm\sqrt{-3}}{2}$$

$$x=\frac{1\pm i\sqrt{3}}{2}$$

So roots are $x=-1,\ \frac{1+i\sqrt{3}}{2},\ \frac{1-i\sqrt{3}}{2}$

1 real 2 imaginary